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Cluster synchronization in star-like complex networks

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Abstract

The cluster synchronization of a star-like complex network which consists of N nodes coupled with a common environment is discussed. By using the matrix theory and the Lyapunov function approach, a sufficient condition about the existence and asymptotic stability of a cluster synchronization invariant manifold is derived. The effectiveness of the sufficient condition is illustrated by two examples. In addition, the two examples also show that, in the star-like complex network in which the individual node is chaotic (or non-chaotic), the dynamics of some nodes in the cluster synchronization invariant manifold can be non-chaotic (or chaotic).

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1. Introduction

Synchronization phenomena have attracted increasing attention from various fields of science and engineering today. Many types of synchronization [1–14] have been introduced and researched: complete or identical synchronization, phase synchronization, cluster synchronization and so on. Recently, the propensity for synchronization has been studied in a complex network with directed and weighted links [4, 5]. In addition, an effective method to determine some possible states of cluster synchronization and to ensure their stability is presented for a given nearest-neighborhood network with zero-flux or periodic boundary conditions in [15, 16]. Moreover, a new general method to determine the stability of complete synchronization in networks with different topologies is proposed in [17]. This method combines the Lyapunov function approach with the graph theoretical reasoning. Based on [17], a novel method [12], which constructs a coupling scheme with cooperative and competitive weight couplings, is used to stabilize arbitrarily selected cluster synchronization patterns with several clusters for connected chaotic networks. In these studies, the master stability function analysis [7] is used to derive a sort of local stability criterion, which is a necessary condition of synchronization; the Lyapunov function approach is used to derive a sort of global stability criterion, which is a sufficient condition of synchronization.

Particularly, cluster synchronization is considered to be significant in communication engineering [13] and biological sciences [14], for example, clustering in the networks consist of coupled cells or functional units which have a complicated nonlinear behavior. Cluster synchronization in networks is observed when the network of oscillators splits into subgroups, called clusters, such that the oscillators within one cluster in synchrony move but the motion of different clusters is not synchronized at all. That is to say, the N nodes in a network split into n clusters, such that $\lim_{t\to\infty} \|\boldsymbol{x}_i(t, \mathbf{x}_0) - \boldsymbol{x}_j(t, \mathbf{x}_0)\| = 0$ holds for the states $\boldsymbol{x}_i(t, \mathbf{x}_0), \boldsymbol{x}_j(t, \mathbf{x}_0)$ corresponding to arbitrary node indices i, j in the same cluster and arbitrary initial states $\mathbf{x}_0 = (\boldsymbol{x}_1(0), \ldots, \boldsymbol{x}_N(0))$ but for arbitrary nodes indices k, l in any two distinct clusters, there exist $\mathbf{y}_0 = (\boldsymbol{y}_1(0), \ldots, \boldsymbol{y}_N(0))$ such that $\lim_{t\to\infty} \|\boldsymbol{x}_k(t, \mathbf{y}_0) - \boldsymbol{x}_l(t, \mathbf{y}_0)\| \neq 0$ holds for the states $\boldsymbol{x}_k(t, \mathbf{y}_0)$ and $\boldsymbol{x}_l(t, \mathbf{y}_0)$. When the number of clusters is 1, cluster synchronization turns into complete synchronization.

Up till now, many studies [15–22] about complete synchronization and cluster synchronization focus on the network consisting of linearly coupled identical systems

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) + \epsilon \sum_{j=1}^{N} c_{ij}(t) P \boldsymbol{x}_j, \qquad i = 1, \ldots, N, \tag{1}$$

where $\boldsymbol{x}_i = \left(\boldsymbol{x}_i^1, \ldots, \boldsymbol{x}_i^m\right)^T$ is the m-dimensional state variable of the ith node and N is the total number of nodes. The nonzero elements of the $m \times m$ matrix P determine the coupling components among the states of nodes. Usually, for simpleness and clarity, $P = \mathrm{diag}(p_1, p_2, \ldots, p_m)$, where $p_h > 0$ for $h = 1, \ldots, s$ and $p_h = 0$ for $h = s+1, \ldots, m$. $\epsilon > 0$ is the coupling strength. The coupled matrix $C = (c_{ij}(t))$ is a $N \times N$ real symmetric matrix reflecting the network topology. In the studies, C is always diffusive (i.e., the sum of all elements in each row is equal to zero), and usually, C is a constant matrix. It is well known that a diffusively coupled matrix has an eigenvalue 0. In an appropriate network topology [12, 15, 22] with diffusive couplings, the eigenvalue 0 of the matrix C can be associated with an eigenvector $(1, 1, \ldots, 1)^T$ or associated with n eigenvectors $\xi_i = (\alpha_{1i}, \ldots, \alpha_{ni})^T$, where $\alpha_{ki} \in R^{m_k}$,

$$\alpha_{ki} = \begin{cases} (1, 1, \ldots, 1), & \text{if} \quad k = i \\ (0, 0, \ldots, 0), & \text{otherwise,} \end{cases}$$

then $M^* = \{\boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N\}$ or $\{\boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_{m_1}, \ldots, \boldsymbol{x}_{m_1+\cdots+m_{n-1}+1} = \cdots = \boldsymbol{x}_N\}$ is an invariant manifold of network (1). Here, the network dynamics in the invariant manifold M^* is $\{\dot{\boldsymbol{x}}_1 = \boldsymbol{f}(\boldsymbol{x}_1), \boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N\}$ or $\left\{\dot{\boldsymbol{x}}_{m_1} = \boldsymbol{f}(\boldsymbol{x}_{m_1}), \boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_{m_1}\right\} \times \left\{\dot{\boldsymbol{x}}_{m_1+m_2} = \boldsymbol{f}(\boldsymbol{x}_{m_1+m_2}), \boldsymbol{x}_{m_1+1} = \cdots = \boldsymbol{x}_{m_1+m_2}\right\} \times \cdots \times \left\{\dot{\boldsymbol{x}}_N = \boldsymbol{f}(\boldsymbol{x}_N), \boldsymbol{x}_{m_1+\cdots+m_{n-1}+1} = \cdots = \boldsymbol{x}_N\right\}$, and obviously, unrelated to the coupling strength ϵ. If the invariant manifold is asymptotically stable under some conditions, network (1) realizes complete synchronization or cluster synchronization.

In this paper, we consider the star-like complex network

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) + \epsilon(-c_i P \boldsymbol{x}_i + \boldsymbol{u}(v(t), \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)), \qquad i = 1, \ldots, N, \tag{2}$$

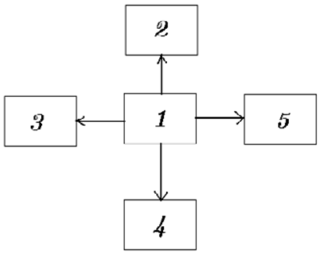

Figure 1. A star network with a hub node.

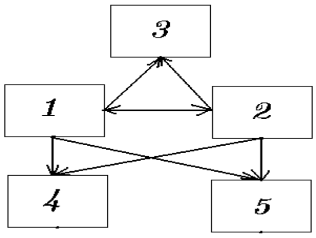

Figure 2. A star network with two hub nodes.

where $\boldsymbol{x}_i$ is the m-dimensional state variable of the ith node and N is the total number of nodes. The nonzero elements of the $m \times m$ matrix P determine the self-coupling components in the states of nodes. Both $\boldsymbol{f}(\bullet)$ and $\boldsymbol{u}(\bullet)$ are vector functions. $\epsilon > 0$ is the coupling strength parameter.

For network (2), $\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i)$ describes the dynamics of the individual node. $\boldsymbol{u}(\bullet)$ can be regarded as an environment state variable which depends on the node states $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ and other influence factor $v(t)$, and then the state variables of any two nodes are indirectly interacted with each other by a common environment state variable $\boldsymbol{u}(\bullet)$. Therefore, network (2) is a star-like network in which $\boldsymbol{u}(\bullet)$ is the state variable of the hub node. $\epsilon(-c_i P\boldsymbol{x}_i + \boldsymbol{u}(v(t), \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N))$ indicates the influences of both the state of the ith node and the environment state on the dynamics of the ith node under coupling actions. It is reasonable that, to a common environment, the response coefficient c_i may be the same as or different from the response coefficient c_j.

Many networks can be described as network (2). For example, all cells in a tissue form a cell network. In the cell network, the dynamics of any individual cell obeys the same laws; all cells interact with a common environment which depends not only on the cells but also on the other influence factors. In the environment, the response coefficient of the ith cell may be the same as or different from the response coefficient of the jth cell. Network (9) in [23] is a special case of network (2).

Obviously, when $\boldsymbol{u}(\bullet) = P\boldsymbol{u}_1(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$, where $\boldsymbol{u}_1(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is a linear function about $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, network (2) is a special case of network (1). Particularly, when $\boldsymbol{u}_1(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is $\boldsymbol{x}_1, \boldsymbol{x}_1 + \boldsymbol{x}_2$ or $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_N$, the corresponding network is a star network with a hub node and unidirectional couplings, a star network with two hub nodes and asymmetrical couplings or a globally coupled network (see figures 1–3 when $N = 5$). In addition, $\boldsymbol{u}(\bullet)$ can be nonlinear also.

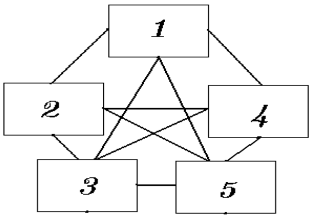

Figure 3. A globally coupled network.

2. Cluster synchronization in the star-like network

First, we give the following hypothesis.

Hypothesis 1. Assume that $f(\bullet)$ satisfies $f(x_j) - f(x_i) = A_{x_i,x_j}(x_j - x_i)$ for a bounded matrix A_{x_i,x_j} with elements depending on x_i and x_j.

Apparently, if $f(\bullet)$ satisfies the Lipschitz condition, then hypothesis 1 holds. Note that many neural networks and chaotic dynamical systems, such as Chua's circuit, satisfy hypothesis 1.

Among $c_i, i = 1, \ldots, N$, there may exist i and j such that $c_i = c_j$. Accordingly, without loss of generality, we assume $c_1 = c_2 = \cdots = c_{m_1}, \ldots, c_{m_1+m_2+\cdots+m_{n-1}+1} = \cdots = c_N$. Therefore, $M^* = \{x_1 = \cdots = x_{m_1}, \ldots, x_{m_1+\cdots+m_{n-1}+1} = \cdots = x_N\}$ is an invariant manifold of network (2). Let $e_{ij} = x_j - x_i$. By (2), for all i, j which satisfy $c_i = c_j$, we have

$$\dot{e}_{ij} = f(x_j) - f(x_i) - \epsilon c_i P e_{ij} = (A_{x_i,x_j} - \epsilon c_i P)e_{ij}. \tag{3}$$

Using the Lyapunov function approach, we obtain the following.

Theorem 1. *Under hypothesis 1, if* $c_1 = c_2 = \cdots = c_{m_1}, \ldots, c_{m_1+m_2+\cdots+m_{n-1}+1} = \cdots = c_N$ *and if there exists a positive definite symmetric matrix* Q *such that*

$$(A_{x_1,x_2} - \epsilon c_i P)^T Q + Q(A_{x_1,x_2} - \epsilon c_i P) < \mathbf{0} \tag{4}$$

holds for arbitrary i, x_1, x_2*, then the invariant manifold* M^* *of network (2) is asymptotically stable.*

Proof. Using (4), for arbitrary i, j,

$$(A_{x_i,x_j} - \epsilon c_i P)^T Q + Q(A_{x_i,x_j} - \epsilon c_i P) < \mathbf{0} \tag{5}$$

holds. Choose the Lyapunov function of the form

$$V = \sum_{c_i=c_j} e_{ij}^T Q e_{ij}.$$

Then, by (5), its time derivative with respect to (3) is

$$\dot{V} = \sum_{c_i=c_j} e_{ij}^T [(A_{x_i,x_j} - \epsilon c_i P)^T Q + Q(A_{x_i,x_j} - \epsilon c_i P)] e_{ij} < 0.$$

Based on the Lyapunov stability theory, (3) is asymptotically stable about zero, i.e., the invariant manifold M^* of network (2) is asymptotically stable. This completes the proof of theorem 1. □

Specially, when Q is an identity matrix I, we have the following.

Corollary 2. *Under hypothesis 1, if* $c_1 = c_2 = \cdots = c_{m_1}, \ldots, c_{m_1+m_2+\cdots+m_{n-1}+1} = \cdots = c_N$ *and if*

$$\left(A^T_{\boldsymbol{x}_1,\boldsymbol{x}_2} + A_{\boldsymbol{x}_1,\boldsymbol{x}_2}\right) - \epsilon c_i(P^T + P) < \mathbf{0} \tag{6}$$

holds for all $i, \boldsymbol{x}_1, \boldsymbol{x}_2$*, then the invariant manifold* M^* *of network (2) is asymptotically stable.*

Remark 1. By theorem 1 and corollary 2, it is easy to see that the matrix P can be non-diagonal, even asymmetrical.

Remark 2. Based on the linear algebra theory, if every $c_i > 0$ and $P^T + P$ is positive definite, then there is ϵ_0 such that (6) holds for $\epsilon > \epsilon_0$.

Remark 3. In the diffusively coupling network (1), the dynamics of the nodes in the invariant manifold M^* is unrelated to the coupling strength ϵ and the coupling matrix C reflecting the network topology, and then the uncoupled dynamics describe the synchronized state. However, in network (2), the dynamics of the nodes in the invariant manifold M^* can depend on the coupling strength ϵ and the environment state variable $\boldsymbol{u}(\bullet)$. Generally speaking, it is very complicated. Here, the uncoupled state does not describe the synchronized dynamics and stable clusters are selected for using the c_i terms.

If the conditions in theorem 1 hold, the N nodes in network (2) split into n clusters and the sets of subscripts of the n clusters are $G_1 = \{1, 2, \ldots, m_1\}, \ldots, G_n = \{m_1 + \cdots + m_{n-1} + 1, \ldots, N\}$, where $N = m_1 + m_2 + \cdots + m_n$. The oscillators within one cluster in perfect synchrony move. Generally, the motion of different clusters is not synchronized. Therefore, cluster synchronization is achieved and the cluster synchronization threshold is related to the response coefficient c_i rather than unrelated to the environment state variable $\boldsymbol{u}(\bullet)$. However, when an invariant submanifold in the invariant manifold M^* is asymptotically stable, the motion of two different clusters may be synchronized, i.e., two different clusters may incorporate a bigger cluster. It is indicated by the following examples.

3. Two examples

In [23], a special case of network (2) ($c_i = c_j$ for any i and j, $\boldsymbol{u}(\bullet)$ depends on all $\boldsymbol{x}_i$) is investigated. Here, two special examples, in which $\boldsymbol{u}(\bullet)$ only depends on $\boldsymbol{x}_1$, are used to illustrate theorem 1 and corollary 2. In example 1, $\boldsymbol{u}(\bullet)$ is linear; but in example 2, $\boldsymbol{u}(\bullet)$ is nonlinear.

Example 1. We consider network (2) which is consisted of N coupled Chua's circuits ($N = 4$). The individual Chua's circuit [24] is described by

$$\begin{aligned} &\dot{x}_i = \alpha(y_i - x_i - g(x_i)), \qquad \dot{y}_i = x_i - y_i + z_i, \\ &\dot{z}_i = -\beta y_i, \qquad i = 1, \ldots, 5, \end{aligned} \tag{7}$$

where $\alpha > 0, \beta > 0$ are parameters, $g(x_i) = bx_i + \frac{1}{2}(a-b)(|x_i+1| - |x_i-1|)$ with $a < b < 0$. When $c_1 = c_2 = 1, c_3 = c_4 = 1.1, \boldsymbol{u}(\bullet) = (0.99x_1, 0.99y_1, 0.99z_1)^T$ and $P = \mathrm{diag}(1, 1, 1)$, network (2) reads

$$\begin{aligned} &\dot{x}_i = \alpha(y_i - x_i - g(x_i)) + 0.99\epsilon x_1 - c_i\epsilon x_i, \\ &\dot{y}_i = x_i - y_i + z_i + 0.99\epsilon y_1 - c_i\epsilon y_i, \\ &\dot{z}_i = -\beta y_i + 0.99\epsilon z_1 - c_i\epsilon z_i, \qquad i = 1, \ldots, 4. \end{aligned} \tag{8}$$

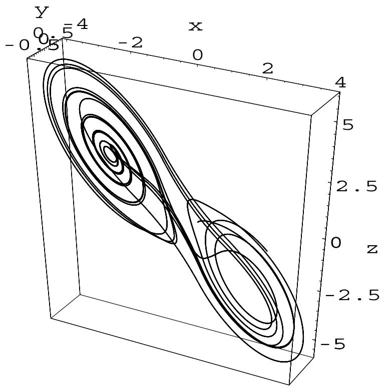

Figure 4. Chaotic attractor in the individual Chua's circuit.

By a simple analysis, we obtain

$$g(x_j) - g(x_i) = l_{x_i,x_j}(x_j - x_i),$$

where l_{x_i,x_j} depends on both x_i and x_j, and $a \leqslant l_{x_i,x_j} \leqslant b$:

$$A_{x_i,x_j} = \begin{pmatrix} -\alpha(1 + l_{x_i,x_j}) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}.$$

Therefore,

$$\left(A^T_{x_i,x_j} + A_{x_i,x_j}\right) - \epsilon c_i (P^T + P) = \begin{pmatrix} -2\alpha\left(1 + l_{x_i,x_j}\right) - 2c_i\epsilon & 1+\alpha & 0 \\ 1+\alpha & -2-2c_i\epsilon & 1-\beta \\ 0 & 1-\beta & -2c_i\epsilon \end{pmatrix}.$$

Based on the well-known Gershgorin disk theorem [25] in the matrix theory, when

$$\epsilon > \epsilon_0 = \max\left\{-\alpha\left(1 + l_{x_i,x_j}\right) + |1+\alpha|/2, -1 + |1-\beta|/2 + |1+\alpha|/2, |1-\beta|/2\right\},$$

condition (6) holds. By corollary 2, the invariant manifold M^* of network (8) is asymptotically stable.

To check the theoretical results, we calculate numerically the errors

$$e_{ij} = \max_{1 \leqslant i < j \leqslant 4} \{|x_j - x_i|, |y_j - y_i|, |z_j - z_i|\}$$

for coupled Chua's circuits, with environment state variable $\boldsymbol{u}(\bullet) = (0.99x_1, 0.99y_1, 0.99z_1)^T$, parameters $\alpha = 10, \beta = 15, a = -1.31, b = -0.75, \epsilon = 11.6 (> \epsilon_0 = 11.5)$ and the random initial conditions $(x_i(0), y_i(0), z_i(0))$ in the region $(-1, 1) \times (-1, 1) \times (-1, 1)$.

By the aid of Mathematica 4, we obtain figures 4–9. Figure 4 corresponds to the chaotic attractor of Chua's circuit when the coupling strength parameter ϵ is equal to zero. Figures 5 and 6 show the time evolution of (x_1, y_1, z_1) and (x_3, y_3, z_3) in network (8) ($\epsilon = 11.6, \boldsymbol{u}(\bullet) = (0.99x_1, 0.99y_1, 0.99z_1)^T, c_1 = c_2 = 1, c_3 = c_4 = 1.1$ and $P = \mathrm{diag}(1, 1, 1)$). In figures 7–9, the evolution of the errors e_{ij} is illustrated. It is very easy to see the errors $e_{ij} \to 0$ when $t \to \infty$ if i and j are the indices in the subscript set of a cluster but the errors $e_{ij} \nrightarrow 0$ when $t \to \infty$ if i and j are the indices in the subscript sets of two different clusters. That is to say, the 2-cluster synchronization in network (8) is realized.

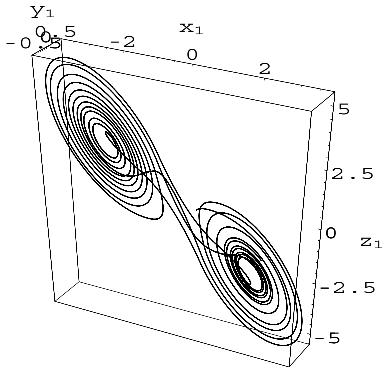

Figure 5. The time evolution of (x_1, y_1, z_1) in network (8).

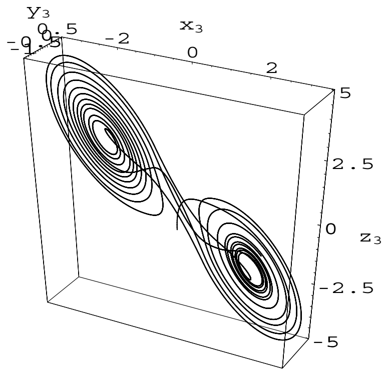

Figure 6. The time evolution of (x_3, y_3, z_3) in network (8).

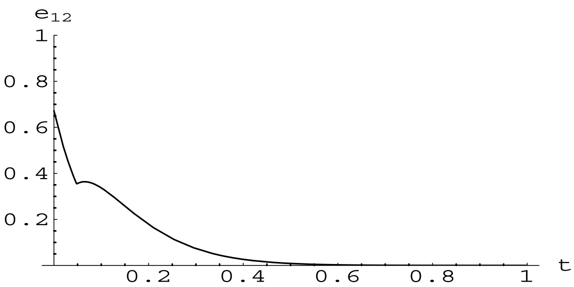

Figure 7. The time evolution of e_{12} when $\epsilon = 11.6$.

On second thoughts, we investigate the synchronization states of network (8) when $\epsilon \geqslant 2000$ and other conditions are retained. It is showed by some simple calculations that network (8) has a globally asymptotic stable equilibrium $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})^T$, where $\mathbf{0} = (0, 0, 0)$. The state of any node (x_i, y_i, z_i) approaches $\mathbf{0}$ when $t \to \infty$. Here, cluster synchronization turns into complete synchronization. In figures 10–12, the evolution of the errors e_{ij} is

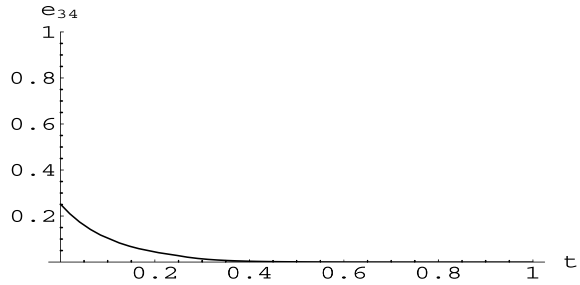

Figure 8. The time evolution of e_{34} when $\epsilon = 11.6$.

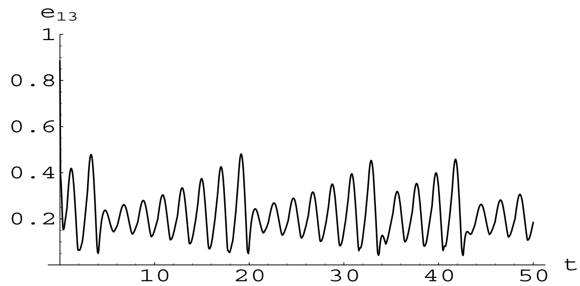

Figure 9. The time evolution of e_{13} when $\epsilon = 11.6$.

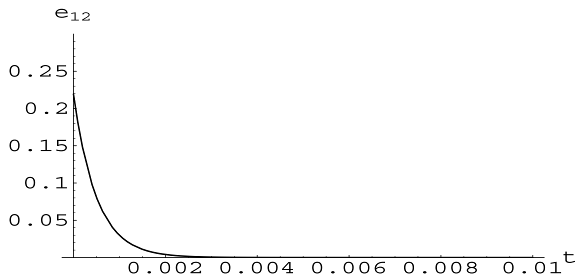

Figure 10. The time evolution of e_{12} when $\epsilon = 2000$.

illustrated for $\epsilon = 2000$ and the random initial conditions $(x_i(0), y_i(0), z_i(0))$ in the region $(-1, 1) \times (-1, 1) \times (-1, 1)$. It is very easy to see the errors $e_{ij} \to 0$ when $t \to \infty$.

The example highlights that, in the star-like network (8) in which the individual node is chaotic, the dynamics of the nodes in the invariant manifold M^* can be chaotic or non-chaotic. It depends on the coupling strength parameter ϵ and the environment state variable $\boldsymbol{u}(\bullet)$ and the response parameter c_i. In addition, two different clusters may incorporate a bigger cluster.

Example 2. We consider network (2) which is consisted of N coupled linear systems ($N = 4$). The individual system is described by

$$\begin{aligned} \dot{x}_i &= 10(y_i - x_i), \qquad \dot{y}_i = 28x_i - y_i, \\ \dot{z}_i &= -\tfrac{8}{3}z_i, \qquad i = 1, \ldots, 4. \end{aligned} \tag{9}$$

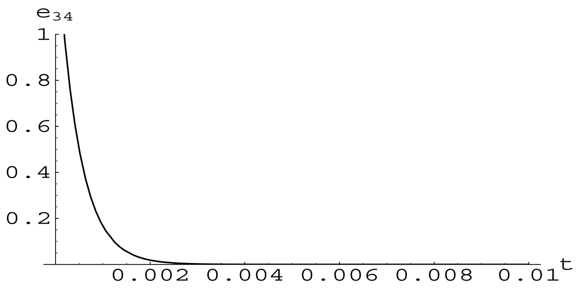

Figure 11. The time evolution of e_{34} when $\epsilon = 2000$.

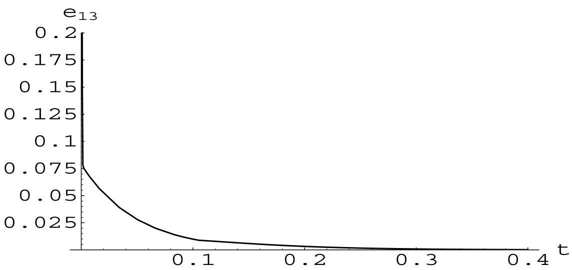

Figure 12. The time evolution of e_{13} when $\epsilon = 2000$.

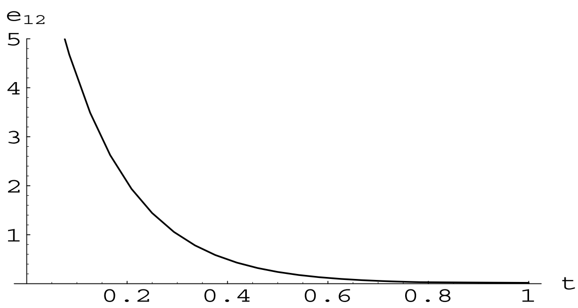

Figure 13. The time evolution of e_{12} when $\epsilon = 19$.

In network (2), $P = \mathrm{diag}(1, 1, 0), c_1 = c_2 = 1, c_3 = c_4 = 1.1$ and $\boldsymbol{u}(\bullet) = (x_1, y_1 - \frac{1}{19}x_1z_1, \frac{1}{19}x_1y_1)^T$. Network (2) reads

$$\dot{x}_i = 10(y_i - x_i) + \epsilon(-c_ix_i + x_1), \qquad \dot{y}_i = 28x_i - y_i + \epsilon(-c_iy_i + y_1) - \tfrac{1}{19}\epsilon x_1z_1,$$
$$\dot{z}_i = -\tfrac{8}{3}z_i + \tfrac{1}{19}\epsilon x_1y_1, \qquad i = 1, \ldots, 4. \tag{10}$$

Because

$$A_{x_i,x_j} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix},$$

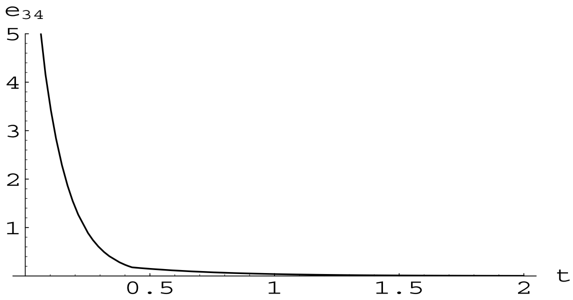

Figure 14. The time evolution of e_{34} when $\epsilon = 19$.

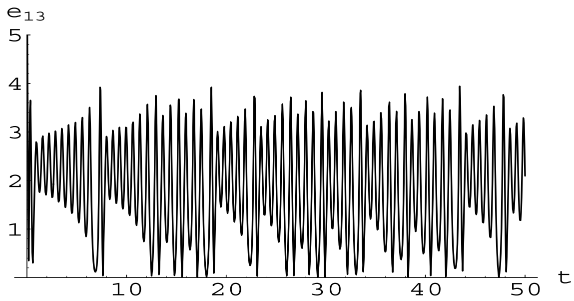

Figure 15. The time evolution of e_{13} when $\epsilon = 19$.

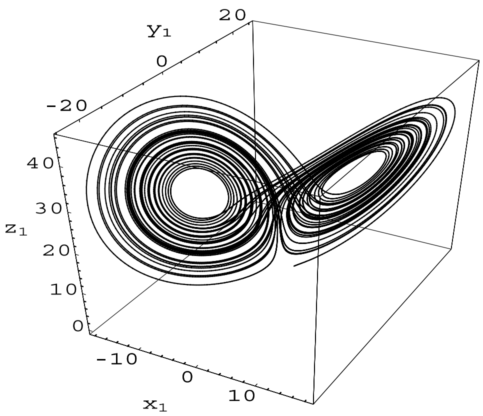

Figure 16. The time evolution of (x_1, y_1, z_1) in network (10).

hypothesis 1 holds. Based on the Gershgorin disk theorem [25], condition (6) holds when $\epsilon > 18$. By corollary 2, the invariant manifold M^* of network (10) is asymptotically stable.

By the aid of Mathematica 4, when $\epsilon = 19$, we obtain figures 13–15 for the random initial conditions $(x_i(0), y_i(0), z_i(0))$ in the region $(0, 10) \times (0, 10) \times (0, 10)$. Figures 16 and 17 show the time evolution of (x_1, y_1, z_1) and (x_3, y_3, z_3) in network (10). In figures 13–15, the evolution of the errors e_{ij} is illustrated. It is very easy to see that, when $t \to \infty$, the errors

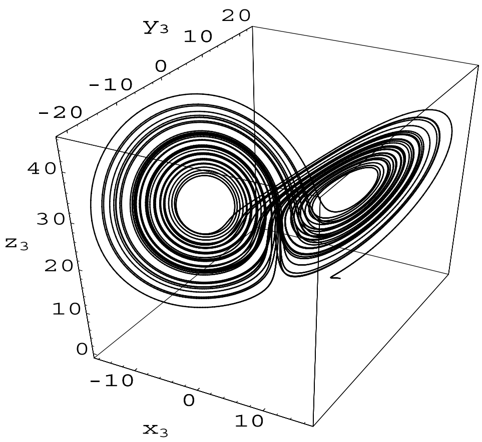

Figure 17. The time evolution of (x_3, y_3, z_3) in network (10).

$e_{12} \to 0$ and $e_{34} \to 0$ but the error $e_{13} \nrightarrow 0$. That is to say, the 2-cluster synchronization in network (10) is realized. The effectiveness of the theoretic results is illustrated.

Apparently, when $\epsilon = 19$, the dynamical equation of the first node in network (10) is the Lorenz system. The example highlights that, in the star-like network (2) in which the individual node is non-chaotic, the dynamics of some nodes in the invariant manifold M^* can be chaotic. Usually, this phenomenon is said to be *emergence*.

4. Conclusion

In this paper, we investigate the cluster synchronization of a star-like network which is consisted of N nodes coupled with a common environment. By the use of the matrix theory and the Lyapunov function approach, a sufficient condition about the existence and stability of a cluster synchronization invariant manifold M^* is obtained. In some cases, an invariant submanifold in the cluster synchronization invariant manifold may be stable. That is to say, two different clusters may incorporate a bigger cluster. Whereas in usual synchronization theory the uncoupled dynamics describe the synchronized state; here the uncoupled state does not describe the synchronized dynamics, which depends on the coupling strength parameter, and stable clusters are selected for using the c_i terms. Then, the effectiveness of our theoretic results is illustrated by two examples. The first example shows that, in the star-like network (2) in which the individual node is chaotic, the dynamics of the nodes in the invariant manifold M^* can be chaotic or non-chaotic. The second example highlights that, in the star-like network (2) in which the individual node is non-chaotic, some nodes in the invariant manifold M^* can emerge chaotic dynamics.

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